The postulates of quantum Mechanics (From Michael A Nielsen)

Postulate 1 (Sets up arena)

Associated to any isolated physical system, is a complex vector space with inner products is known as state space of the system. The state space is a subset of Hilbert Space.

The system is completely described by the state vector $|\Psi\rangle$ which is a unit vector in system's state space / Hilbert's space.

Quantum Mechanics doesnot tell the state space & state vector

The condition that $|\Psi\rangle$ is a unit vector means

$$\boxed{\langle \Psi | \Psi \rangle = 1}$$

Postulate 2 (Time evolution)

The evolution of a closed system / isolated system is described by a unitary transformation.

$$\boxed{|\Psi(t_2)\rangle = U(t_1, t_2)\,|\Psi(t_1)\rangle}$$

Quantum Mechanics doesn't tell which ~~matrices~~ unitary operators to use.

This is the discrete-time description of dynamics which depends on only t_1 and t_2

For continous time description, Hamiltonians are used. Thus the second postulate becomes ②

[The time evolution of the state of a quantum system is described by Schrodinger equation.

$$\boxed{i\hbar \frac{d|\Psi\rangle}{dt} = \hat{H}|\Psi\rangle}$$
]

If we know the hamiltonian, we know the dynamics of the system completely.

Hamiltonian can be time dependent or independent, depending on physical condition. If the conditions are changing with time (eg oscillating electric field) then Hamiltonian is time dependent

Because Operator $\hat{H}$ is hermitian, thus it's spectral decomposition can be written as

$$\boxed{\hat{H} = \sum_i E_i |E_i\rangle\langle E_i|}$$

$|E_i\rangle$ orthonormal basis
E_i eigen value w.r.t basis

$|E_i\rangle$ are energy eigen vectors and E_i is their eigen value

variation of $|E_i\rangle$ w.r.t time is given by

$$\boxed{|E_i(t)\rangle = e^{-\frac{iE_it}{\hbar}} |E_i(t=0)\rangle}$$

From schrodinger's equation and unitary transformation equation, one can verify that

$$U(t_1, t_2) \equiv e^{-iH(t_2-t_1)/\hbar}$$ (Expand $e^{-i\theta}$)

Postulate 3 (Measurement)

Quantum measurements are described by a collection $\{M_m\}$ of measurement operators. These operators act on the state space of the system being measured. 'm' indicates the total number of measurement outcomes that may occur in the experiment.

If the state vector of a system was $|\psi\rangle$ before measurement then the probability that result 'm' occurs is

$$\boxed{p(m) = \langle \psi | M_m^{+} M_m | \psi \rangle}$$

And the state $|\psi\rangle$ after measurement changes to

$$|Q\rangle = \frac{M_m |\psi\rangle}{\sqrt{\langle \psi | M_m^{+} M_m | \psi \rangle}}$$

There is certain restriction over measurement operators

$$\boxed{\sum_i M_i^{\dagger} M_i = I} \quad \text{(completeness equation)}$$

completeness equation ensures that $\boxed{\sum_i p(i) = 1}$

Consider $\{M_m\}$ as a set of operators (measurement of course)

thus according to the postulate $M_m : H \longrightarrow H$ (H is hilbert space of system)

$$M_m = |\alpha_j\rangle\langle\beta_k|$$

where $|\alpha_j\rangle, |\beta_k\rangle$ belong to vector space of system

For example

consider system of one qubit.

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

Let us define measurement operator as $M_0 = |0\rangle\langle 0|$, $M_1 = |1\rangle\langle 1|$

thus $p(0) = \langle \psi | M^{\dagger} M | \psi \rangle$

$$= |a|^2$$

and after the measurement

$$|\psi\rangle = \frac{a}{|a|}|0\rangle$$

<u>Distinguishing Quantum states</u>

Consider two persons Alice and Bob. Let Alice produce two orthonormal states $|\psi_1\rangle$ and $|\psi_2\rangle$. The job of bob is to find or rather distinguish $|\psi_1\rangle, |\psi_2\rangle$.

Let Alice send two states to bob $|\psi_a\rangle$ and $|\psi_b\rangle$. Only Alice knows the true identity.

Bob makes two <u>Measurement operators</u> $M_1 = |\psi_1\rangle\langle\psi_1|$ and $M_2 = |\psi_2\rangle\langle\psi_2|$ such that $M_1 + M_2 = I$ (Completeness)

Bob then operates M_1 on $|\psi_a\rangle$ and if he gets the result 1 then $|\psi_a\rangle = 1$ $[\,p(1) = \langle\psi_a|\psi_1\rangle\langle\psi_1|\psi_a\rangle = 1\,]$ with certainty

$$\boxed{p(1) = \langle\psi_a|M_1^{\dagger}M_1|\psi_a\rangle \qquad p(2) = \langle\psi_a|M_2^{\dagger}M_2|\psi_a\rangle}$$

If $p(1)$ is zero then $p(2) = 1$ and vice versa. Thus states can be distinguished.

Demonstration of how non orthogonal states are non distinguishable.

Lets consider the qubits. These base staes of qubits are $|0\rangle$ and $|1\rangle$. $\langle 0|1\rangle = 0$

~~Case I~~ ~~(States care~~

Define a set of Measurement operators $M_0 = |0\rangle\langle 0|$ and $M_1 = |1\rangle\langle 1|$. $M_0 + M_1 = I$ and M_0, M_1 are Hermitian

The operation of operators on base states is tabulated

operation	observed value (say)
i) M_0 on $\|0\rangle$	a
ii) M_0 on $\|1\rangle$	no result
iii) M_1 on $\|0\rangle$	no result
iv) M_1 on $\|1\rangle$	b

Case I

Let the States be orthogonal. $|\psi_1\rangle = |0\rangle$ $|\psi_2\rangle = |1\rangle$

We have two sub cases

		result	probability
a)	$M_0 \|\psi_1\rangle = \|0\rangle$	a	$\langle 0\|M_0^{\dagger} M_0\|0\rangle = 1$
	$M_0 \|\psi_2\rangle = \times$	no result	0 (This case will not happen)
	$M_1 \|\psi_2\rangle = \|1\rangle$	b	$\langle 1\|M_1^{\dagger} M_1\|0\rangle = 1$

Thus the Measurement operators $\{M_0, M_1\}$ when applied on the arbitrary state $|\psi_i\rangle$ give definite result. Thus by seeing the table bob can determine which state it was by seeing the value (a or b).

Another sub case could have been

operation	result	probability

b)

operation	probability	result
$M_0\|\psi_1\rangle$	0	no result

∴ another subcase is ~~not~~ possible just draw it

Case II

(non orthogonal states)

$|\psi_1\rangle = |0\rangle$ $\quad \cancel{|\psi_2\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}}$ $\quad |\psi_2\rangle = \frac{a|0\rangle + b|1\rangle}{(a^2+b^2)}$

again lets consider a sub case

a)

Operation	Probability	result
$M_0\|\psi_1\rangle$	$\langle 0\|M_0^\dagger M_0\|0\rangle = 1$	a
i) $M_0\|\psi_2\rangle$	$\langle \psi_2\|M_0^\dagger M_0\|\psi_2\rangle = 1/2$	a
ii) $M_1\|\psi_2\rangle$	$\langle \psi_2\|M_1^\dagger M_1\|\psi_2\rangle = 1/2$	b

~~or he could have taken $|\psi_2\rangle$ first~~

~~b) operation probability result~~

~~$M_0|\psi_2\rangle$ 1/2 a~~

~~$M_1|\psi_2\rangle$~~

operation	Probability	result
$M_0\|\psi_1\rangle$	$\langle 0\|M_0^\dagger M_0\|0\rangle = 1$	a
$M_1\|\psi_1\rangle$	$\langle 0\|M_1^\dagger M_1\|0\rangle = 0$	no result
$M_0\|\psi_2\rangle$	$\langle \psi_2\|M_0^\dagger M_0\|\psi_2\rangle = 1/2$	a
$M_1\|\psi_2\rangle$	$\langle \psi_2\|M_1^\dagger M_1\|\psi_2\rangle = 1/2$	b

But if $|\psi_1\rangle$ and $|\psi_2\rangle$ are not orthogonal, then $p(1), p(2)$ won't be non zero simultaneously. ⑤

Theorem: Non orthogonal states cant be distinguised.

Proof: Let $|\psi_1\rangle$ and $|\psi_2\rangle$ be non orthogonal states. Lets, ~~then~~, assume that they are distinguishable.

The measurement operator $M_i = |\psi_i\rangle\langle\psi_i|$

Completeness equation $\sum M_i^2 = I$ ~~is satisfied~~

Define $\quad E_i = \sum_{j:\, f(j)=i} M_j^{\dagger} M_j \quad$ ($f(j)$ is a rule where j is the result corresponding to ith measurement)

such that $\sum_i E_i = I$

thus $\quad \langle\psi_1|E_1|\psi_1\rangle = \langle\psi_2|E_2|\psi_2\rangle = 1$

also $\quad \langle\psi_1|E_2|\psi_1\rangle = \langle\psi_2|E_1|\psi_2\rangle = 0$

i.e $\quad \sqrt{E_2}\,|\psi_1\rangle = 0$

Let $\quad |\psi_1\rangle = |\psi_1\rangle \qquad$ ($|\psi_n\rangle$ is normal to $|\psi_1\rangle$)

$\quad |\psi_2\rangle = \alpha|\psi_1\rangle + \beta|\psi_n\rangle \qquad (|\beta|^2 < 1)$

$\therefore \sqrt{E_2}\,|\psi_2\rangle = \beta\sqrt{E_2}\,|\psi_n\rangle$

$$\langle\psi_2|E_2|\psi_2\rangle = |\beta|^2\langle\psi_n|E_2|\psi_n\rangle \le |\beta|^2 < 1$$

$$\langle\psi_n|E_2|\psi_n\rangle \le \sum_i \langle\psi_n|E_i|\psi_n\rangle = 1$$

Special case of postulate 3 "Projective measurements"

A projective measurement is described by an observable M, which is hermitian operator on a state space of the system being observed. The spectral decomposition of observable M is

$$\boxed{M = \sum_m m P_m}$$

where P_m is a projector onto the eigen space of M with eigen value m. The ~~total~~ ~~number~~ of possible outcomes of the measurement corresponds to eigen value m.

On measuring the state $|\psi\rangle$ by m operator leads to

$$|\alpha\rangle = \frac{P_m |\psi\rangle}{\sqrt{p(m)}}$$

$$p_m = \langle \psi | P_m^{\dagger} P_m | \psi \rangle$$
$$= \langle \psi | P_m | \psi \rangle$$

Note: Here observable has been defined. All observables are measurement operators but all measurement operators are not observables

If M_m of postulate 3 is converted into orthogonal projectors then we will be having this situation

Expectation value is a statistical concept defined by

$$E(M) = \sum_m m\, p(m)$$
$$= \sum_m m \langle \psi | P_m | \psi \rangle = \langle \psi | M | \psi \rangle$$
$$= \langle M \rangle$$

The generally used conventions for Postulate 3

rather Projective Measurements:

a) Rather than giving an observable (Hermitian operator) simply a complete set of orthogonal projectors satisfying completeness equation $\sum_m P_m = I$ and $P_m P_{m'} = \delta_{mm'} P_m$ is listed. It simply means $\boxed{M = \sum_m m P_m}$

b) Phrase "to measure in basis $|m\rangle$" means that the Projectors P is onto ~~orthonormal basis $|m\rangle$~~ the subspace of orthonormal basis $|m\rangle$. ~~[illegible]~~

means $\boxed{P_m = |m\rangle\langle m|}$

[Since M is a diagonal matrix with eigen vectors $|m\rangle$ which are forming the basis of subspace projected by P_m one can also say that the eigen vectors of Hermitian operator form basis]

Example $\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ or $\hat{\sigma_z} = |0\rangle\langle 0| - |1\rangle\langle 1|$

Thus the measurement of Z (observable) on state $|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$

will give state $|0\rangle$ with 1/2 and state $|1\rangle$ with 1/2

probability

More generally if $\bar{v}$ is any real three dimensional unit vector then an observable (spin) is described by

$$\bar{V} \cdot \bar{\sigma} = V_x \sigma_x + V_y \sigma_y + V_z \sigma_z$$

$$\bar{B} \cdot \bar{\sigma} = B_x \sigma_x + B_y \sigma_y + B_z \sigma_z$$ (see Feynman lectures notes)

(Also known as measurement of spin along $\bar{v}$ axis)