Linear Algebra

$V(F)$ is a vector space over the field F

Field :- Field is a ring whose nonzero elements form commutative group.

Ring is an algebric structure (consisting a set together with two binary operation (addition and multiplication)) ring axiom require addition is commutative, addition multiplication are associative, distributive.

* consider $W(F)$

Point 1

Let $S = \{ |w_1\rangle, |w_2\rangle \text{ ----- } |w_d\rangle \}$ be subset of W.

S is said to be linearly dependent if there exists scalars a_i with $a_i \neq 0$ for atleast one i such that

$$\boxed{\sum_{i=1}^{d} a_i |w_i\rangle = 0} \quad \text{— (1)}$$

If (1) holds true if $a_i = 0 \ \forall i$ then S is linearly independent subset of W

Example

a) $W = \mathbb{R}^3$

Let $S = \{(a, b, c)\}$ atleast one of a, b, c is non zero

S is linearly independent subset of W

$$\boxed{\because \ c_1(a,b,c) = (0,0,0) \Rightarrow c_1 = 0}$$

dimensions should be same

b) $W = \mathbb{R}$

$S = \{a, b\}$ $\quad a \neq 0 \quad b \neq 0$

$$c_1 a + c_2 b = 0$$

$$\boxed{c_1 = -c_2 \frac{b}{a}} \quad \therefore S \text{ is linearly dependent}$$

c) $V = \mathbb{R}^2$

$S = \{U, V\}$

$U = (a,0)$; $V = (0, b)$

$c_1 (a,0) + c_2 (0,b) = (0,0)$

$(c_1 a, c_2 b) = (0,0)$

$c_1 = c_2 = 0$

$\therefore$ S is linearly independent

d) $V = \mathbb{R}^2$

$S = \{U, V\}$

$U = (1,1)$ $V = (1,0)$

$(c_1 + c_2, c_1) = (0,0)$

$c_1 + c_2 = 0$

$c_1 = 0$

L.I

Point 2

$W(F)$ is a vector space

$|\omega_1\rangle$ - - - - $|\omega_d\rangle$ be any elements $\in W$

a_1 - - - - - a_d be any scalars $\in F$

$\sum_{i=1}^{d} a_i |\omega_i\rangle$ = linear combination of $|\omega_1\rangle \cdots |\omega_d\rangle$

Linear span, → collection (set) of all linear combinations

let S be a linear span of W then

$S = \{ |p\rangle, |q\rangle \cdots \}$

where $|p\rangle = \sum_{i=1}^{d} a_i |\omega_i\rangle$ $|q\rangle = \sum_{i=1}^{d} b_i |\omega_i\rangle$ and so on

Theorem : linear span s forms subspace of W

Proof can be seen by adding p and q and saying

$a_i + b_i \in F$ $p+q \in S$

Point 3

~~Basis~~ : Let $W(F)$ be a vector space

Let $S = \{ |\omega_1\rangle, |\omega_2\rangle ------ |\omega_d\rangle \}$ be subset of W

Then S is said to be a basis of W if

① S is a linearly independent subset of W ✓

② Linear span of S must be equal to W ✓

ie ① $\sum_{i=1}^{d} a_i |\omega_i\rangle = 0$ if and only if $a_i = 0 \ \forall i$

② I can create all other elements of vectors space W by some linear combination of the elements of S

If S is a basis, then elements can be called as basis vectors/elements

example (for real fields)

a) $W = \mathbb{R}$ $\quad S = \{\alpha_0$ where $\alpha_0 \in W$ & $\alpha_0 \neq 0\}$

S is a basis

b) $W = \mathbb{R}^2$, $F = \mathbb{R}$ $\quad S = \{ (1,1), (1,0) \}$

i) $(C_1 + C_2, C_1) = (0,0)$

$\boxed{C_1 = C_2 = 0}$

ii) also $(C_1 + C_2, C_1) \in \mathbb{R}^2$ if $C_1, C_2 \in \mathbb{R}$

ie linear span $\boxed{\omega \subseteq \mathbb{R}^2}$ (ω is subset of $\mathbb{R}^2$)

~~Also / vector space $W / \in \mathbb{R}$~~

for any element $(a, b) \in \mathbb{R}^2$

$(a, b) = (C_1 + C_2, C_1)$ $\quad C_1 = b \quad C_2 = a - b$

$(a, b) = b(1,1) + (a-b)(1,0)$ $\ \in \omega$

$\boxed{\mathbb{R}^2 \subset \omega}$

Point 4

Let there be two vector spaces $V(F)$ & $W(F)$. Consider the mapping T from V onto W such that for any $|V_1\rangle, |V_2\rangle \in V$ $C \in F$

T is linear transformation if

$$T(C|V_1\rangle + |V_2\rangle) = C\,T(|V_1\rangle) + T(|V_2\rangle)$$

two varieties
a) Dont change the vectors
b) Change the vectors themselves

where $T(|V_1\rangle) = |W_1\rangle \in W$
$T(|V_2\rangle) = |W_2\rangle$

if $V = W$ then T is linear operator on V

[This structure associates each pair of vectors with a scalar quantity. Intuitive geometrical notations such as length of vector, angle between the vectors]

Point 5 (structure adding.)

Consider $V(F)$. The innerproduct defined in V is a scalar valued function which maps every ordered pair vectors of V into a scalar in F such that following conditions are satisfied. (For example consider trace for matrices vectors integration for continous function vector)

For every $\alpha, \beta, \gamma \in V$ & $C \in F$ (Note: we demand these conditions for application purposes)

① $\langle \alpha+\beta, \gamma\rangle = \langle\alpha,\gamma\rangle + \langle\beta,\gamma\rangle$ [linearity]

② $\langle C\alpha, \beta\rangle = C\langle\alpha,\beta\rangle$

③ $\langle\alpha,\beta\rangle = \overline{\langle\beta,\alpha\rangle}$ complex conjugate (conjugate symmetry)

④ $\langle\alpha,\alpha\rangle \geq 0$ for $\alpha \neq 0$ (positive definiteness)

⑤ $\langle\alpha, C\beta+\gamma\rangle = \bar{C}\langle\alpha,\beta\rangle + \langle\alpha,\gamma\rangle$

Note: vector space equipped with inner product is inner product space. Hilbert space is inner product space with field F as complex.

A linear functional or linear form is a linear map or linear transformation from a vector space $V(F)$ to its field of scalars $\in F$.

[For example in R^n, if you consider vectors as column vectors then functionals are row vectors. (considering mapping as matrix multiplication)]

The set of all linear functionals from $V(F)$ to F is itself a vector space over F. This space is called the dual space of ~~of~~ $V(F)$ denoted by $V^D(F)$

* For every linear Vector Space there exists a unique dual
* For $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^n$ the Dual LVS is itself.

consider a $V(F) = \{\psi_1, \psi_2 \ldots \}$

"consider a special vector ψ_1 and associate a scalar with the ordered pair $[(\psi_1, \psi_i \in V) = \text{scalar}]$"

$(\psi_1, \psi_i \in V) = S_{\psi_1}[\psi_i]$ such that the properties of IPS are satisfied

similarly

$(\psi_2, \psi_i \in V) = S_{\psi_2}[\psi_i]$

linear functionals

Here ψ_1, ψ_2 are mapping ψ_i's to ~~a~~ scalar $\in \mathbb{F}$

~~&~~ $\psi_1, \psi_2 \in V^D$ and denoted by $\langle\psi_1|$, $\langle\psi_2|$

It can be seen that the postulates for ~~$\mathbb{C}^N$~~ Inner product suggest us to take row and column matrices and do the required algebra

a) consider $\mathbb{R}^n$ $\mathbb{F} \in \mathbb{C}$

$$\left.\begin{aligned} |\alpha\rangle &= (x_1, x_2 \cdots x_n) \\ |\beta\rangle &= (y_1, y_2 \cdots y_n) \end{aligned}\right\} \text{(row representation)}$$

$$\langle\, |\alpha\rangle, |\beta\rangle \,\rangle = \langle\alpha|\,|\beta\rangle = \langle\alpha|\beta\rangle \equiv (x_1^*, x_2^* \cdots x_n^*)\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$= \sum_{i=1}^{n} \bar{x}_i y_i$$

~~$\langle\alpha|\alpha$~~

$$\langle\alpha|\alpha\rangle = \sum_{i=1}^{n} |x_i|^2 = (\text{length})^2$$

Now for infinite dimensional LVS

$$\langle\alpha|\alpha\rangle = \sum_{i=1}^{\infty} |x_i|^2 < \infty \qquad \text{is demanded}$$

$$\boxed{\text{Called LVS of square summable called } l_2}$$

for Point 6 (The norm)

one can define $||\alpha|| = \left(\sum_{i}^{\infty} |x_i|^p\right)^{1/p}$ then the vector space will be l_p and its dual $\tilde{l}_p = l_q$ where

$$\frac{1}{p} + \frac{1}{q} = 1$$

Note: only for $p = 2$ q is 2 and dual ~~of~~ LVS is same interpretation demands Square summable besides Probabilistic spaces

For example

(a) $V = \mathbb{R}^n$ $\qquad F \in R$ $\qquad V = \mathbb{C}^n$

$\langle \alpha, \beta \rangle = \sum_{i=1}^{n} x_i \overline{y_i}$

$\alpha = (x_1, x_2 \cdots x_n)$

$\beta = (y_1, y_2 \cdots y_n)$

$\langle \alpha, \beta \rangle = \sum_{i=1}^{n} x_i y_i$

(b) Let V be a space of all complex valued contionus function[0,1]

$V = C[0,1]$

Let f and g be any two elements (vectors) $\in V$

$\langle f, g \rangle = \int_0^1 \bar{f}(x)\, g(x)\, dx$

Point 6

~~Norm ~~is~~ defined as a scalar valued ~~funct~~ function on a vector space V(F) defined as For any $\alpha \in V$ the (+) square root of $\langle \alpha, \alpha \rangle$ is known as norm denoted by $\|\alpha\|$~~

Point 7

we consider a vector $|v_i\rangle \in V(F)$ as ~~is~~ normalised if $\| |v_i\rangle \| = 1$. In quantum mechanics we use the basis which are ortho normal

For example

Consider V(F)

$S = \{ |v_1\rangle, \cdots |v_d\rangle \}$ represent the basis (ie LI and LS forms vector space) then by using Gram-Schmidt procedure we create another set B from the elements of S such that B contains orthonormal Basis vectors

$$B = \{ |e_1\rangle, |e_2\rangle ---- |e_d\rangle \}$$

$$S = \{ |v_1\rangle, |v_2\rangle --- |v_d\rangle \}$$

$$\left[|e_{k+1}\rangle = \frac{|v_{k+1}\rangle - \sum_{i=1}^{k} \langle e_i | v_{k+1}\rangle |e_i\rangle}{\left\| |v_{k+1}\rangle - \sum_{i=1}^{k} \langle e_i | v_{k+1}\rangle |e_i\rangle \right\|} \right] \text{ for } 1 \le k \le d$$

$$|e_1\rangle = \frac{|v_1\rangle}{\| |v_1\rangle \|}$$

Revise point 4

Point 8

Matrix representation of linear transform

Let T be a L.T from $V(F) \longrightarrow W(F)$ where

dim. V (number of basis) $= n$ and $\dim(W) = m$

Let

$$B_V = \{ |e_1\rangle ---- |e_d\rangle \}$$ be orthonormal basis of V

$$B_W = \{ |\omega_1\rangle ----- |\omega_d\rangle \}$$ " " " " W

$$(|\omega_i\rangle) = \sum_{j=1}^{n} A_{ij} |v_j\rangle \quad (|v\rangle \text{ is transformed to } \omega) \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_m \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & & & \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$\equiv T(|\alpha\rangle_V)$

Plugin any vector you want

$$|\alpha\rangle_V = \sum_{k=1}^{n} \lambda_k |v_k\rangle$$

$$T(|\alpha\rangle_V) = |\alpha\rangle_W \quad \text{thus} \quad \boxed{T(|\alpha\rangle_V) = A|\alpha\rangle_V}$$

$$A|\alpha\rangle_V = |\alpha\rangle_W$$

$$[T(\alpha)]_{B_W} = AX = A[\alpha]_{B_V}$$

Here A is known as matrix representation of T relative to orthogonal basis B_V and B_W. A is an $m \times n$ matrix

In Quantum Mechanics $A|V_i\rangle$ is a linear transformation from $V(F) \rightarrow V(F)$. It is called operator

Also V, W, X be the vector spaces defined over the field F

$A: V \rightarrow W$ and $B: W \rightarrow X$ are linear then we use BA to denote composition of B with A defined by $BA|V\rangle$.

So what is an operator. Consider the rotation matrix and ~~asse~~ It is an operator which ~~to~~ maps vector from one space to rotated space. But an operator can also change a vector

Point 9

The linear transform can also be ~~treate~~ represented by $A \equiv |x\rangle\langle a|$ where $|x\rangle \in V(F)$ $\langle a| \in V^D$

Thus $A|\psi\rangle = |x\rangle\langle a|\psi\rangle = \langle a|\psi\rangle|x\rangle$ (another vector)

if $|x\rangle$ column vector $\langle a|$ is a row vector then ~~the~~ $|x\rangle\langle a|$ is a matrix itself (Matrix mechanics)

Now $|e_i\rangle\langle e_i|$ (Provided $|e_i\rangle$ is orthonormal bases)
is a special type of operator known as Projection operator.

Lets say
$$|\psi\rangle = \sum_{i=1}^{n} \lambda_i |e_i\rangle$$

now operate $P_3 = |e_3\rangle\langle e_3|$ on $|\psi\rangle$

$$\boxed{P_3 |\psi\rangle = \lambda_3 |e_3\rangle}$$ i.e component of $|\psi\rangle$ along $|e_3\rangle$

$$\boxed{P_i^2 = P_i}$$

There are two important properties of orthonormal bases

* $\langle e_i | e_j \rangle = \delta_{ij}$ (orthonormality)
* $\sum_i |e_i\rangle\langle e_i| = \mathbb{1}$ (completeness)

→ try applying on any general vector $|\psi\rangle$ and you will get $|\psi\rangle$ again

consider $V(F)$ $\quad B = \{|e_1\rangle, |e_2\rangle\}$
it has can be shown that $|e_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|e_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

now $\langle e_i | e_j \rangle = \delta_{ij}$

$|e_1\rangle\langle e_1| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad |e_1\rangle\langle e_2| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad |e_2\rangle\langle e_1| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$|e_2\rangle\langle e_2| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

thus any matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_{12}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + a_{21}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + a_{22}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$A = \sum_{ij} a_{ij} |e_i\rangle\langle e_j|$$

Matrix Element

$\longrightarrow \langle e_i | A | e_j \rangle = a_{ij}$

$|e_i\rangle\langle e_j|$ form the basis of operators

$$A = \sum_{ij} a_{ij} |e_i\rangle\langle e_j|$$

$a_{ij} = \lambda_i \delta_{ij}$ then we get diagonalisable matrix

<u>Applications</u>

a) Allows us to write an operator in any set of desirable basis ~~of~~ / ~~general~~ (completeness)

eg $A: V(F) \to V(F)$ and $B = \{|v_i\rangle\}$

$$A = \mathbb{1} A \mathbb{1}$$

$$= \sum_{ij} |v_i\rangle\langle v_i| A |v_j\rangle\langle v_j|$$

$$= \sum_{ij} \langle v_i | A | v_j\rangle \, |v_i\rangle\langle v_j|$$

$A: V(F) \to W(F)$

$$A = \sum_{ij} \langle w_i | A | v_j \rangle |w_i\rangle\langle v_j|$$

b) Cauchy Schwarz inequality

$$\langle v|v\rangle\langle w|w\rangle \geq |\langle v|w\rangle|^2$$

<u>Proof</u>

Let $|e_i\rangle$ be set of orthonormal basis with $|e_1\rangle = \frac{|w\rangle}{\|w\|}$

$$\sum_i \langle v|e_i\rangle\langle e_i|v\rangle \langle w|w\rangle \geq \frac{\langle v|w\rangle\langle w|v\rangle}{\langle w|w\rangle} \langle w|w\rangle$$

$$= |\langle v|w\rangle|^2$$

Point 10

consider a vector space $V(F)$. A scalar $C \in F$ is called eigen value ~~corresponding~~ of an existing linear transform on $V(F)$, if there exists a nonzero vector $\alpha \in V(F)$ such that

$$\boxed{T(\alpha) = C\alpha}$$

{ T is an operator (L.T)
α is an Eigen vector w.r.t operator
C is eigen value w.r.t α eigen vector

In Quantum Mechanics

$$\hat{H}|n\rangle = E_n|n\rangle$$

Eigenspace : Eigen space corresponding to an eigen value c is a set of vectors having same eigen value C
Eigenspace is subspace of $V(F)$.
In Quantum Mechanics we have degenrate levels

Diagonalisable operators : Operators having diagonal representation

Relation of different representation of operator A:

Consider $\hat{A} : V(F) \longrightarrow V(F)$ [A over V(F)]

[consider Hilbert space]

Consider basis $B_1 = \{|e_1\rangle, |e_2\rangle, |e_3\rangle\}$

Consider a matrix $\boxed{A(|e_i\rangle) = \sum_j A_{ji} |e_j\rangle}$

$$\hat{A} \equiv \begin{array}{c|ccc} & |e_1\rangle & |e_2\rangle & |e_3\rangle \\ \hline \langle e_1| & a_1 & 0 & 0 \\ \langle e_2| & 0 & a_2 & 0 \\ \langle e_3| & 0 & 0 & a_3 \end{array}$$

Diagonalisable matrix

Consider another basis $B_2 = \{|e_1'\rangle, |e_2'\rangle, |e_3'\rangle\}$

such that $T : B_1(F) \longrightarrow B_2(F)$ [Transformation ~~operator~~ operator]

$$T \equiv \begin{array}{c|ccc} & |e_1'\rangle & |e_2'\rangle & |e_3'\rangle \\ \hline \langle e_1| & T_{11} & T_{12} & T_{13} \\ \langle e_2| & T_{21} & T_{22} & T_{23} \\ \langle e_3| & T_{31} & T_{32} & T_{33} \end{array}$$

$$T = \sum_{ij} T_{ij} |e_j'\rangle\langle e_i|$$

But now $\hat{A}$ as matrix is

$$\hat{A} = \begin{array}{c|ccc} & |e_1'\rangle & |e_2'\rangle & |e_3'\rangle \\ \hline \langle e_1'| & A_{11} & A_{12} & A_{13} \\ \langle e_2'| & A_{21} & A_{22} & A_{23} \\ \langle e_3'| & A_{31} & A_{32} & A_{33} \end{array}$$

The operator sure looks different but the characteristic function is same. example translation matrix

consider $B_1 = \{|e_1\rangle, |e_2\rangle, |e_3\rangle\}$

$$A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$$

$$A = \sum_i a_i |e_i\rangle\langle e_i|$$

By definition eigen value / eigen vector w.r.t A can be found using equation

$$\hat{A}\,|\alpha_i\rangle = \lambda_i\,|\alpha_i\rangle$$

One can see if $A = \sum_i a_i |e_i\rangle\langle e_i|$ then $|\alpha_i\rangle = |e_i\rangle$ (A is diagonal)

and $\lambda_i = a_i$ (orthonormal basis are eigen vector of A is diagonal)

Now consider Matrix representation

$AX = \lambda X$ → Eigen vector equation for matrix

$$\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{vmatrix} (a_1-\lambda) & 0 & 0 \\ 0 & (a_2-\lambda) & 0 \\ 0 & 0 & (a_3-\lambda) \end{vmatrix} = 0$$

$$(\lambda - a_1)(\lambda - a_2)(\lambda - a_3) = 0$$

$\lambda = a_1, a_2, a_3 \equiv$ Eigen value

also $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ corresp. to a_1 $\quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ corresp to a_2 $\quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ corresp to a_3

thus $|e_1\rangle \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\quad |e_2\rangle \equiv \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\quad |e_3\rangle \equiv \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

in matrices

Point 12

· Definitions

a) $A : V(F) \longrightarrow V(F)$

then there exists an unique linear transform $A^{\dagger}$ such that

$$\langle V | A | W \rangle^{*} = \langle W | A^{\dagger} | V \rangle$$

defined by Hermitian conjugate

b) Hermitian operators $A^{\dagger} = A$

c) ~~Proje~~ Consider $P : V(F) \longrightarrow W(F)$ where $W \subseteq V$

$B_V = \{ |1\rangle, |2\rangle \dots |d\rangle \}$

W ~~B~~ $= \{ |1\rangle, |2\rangle \dots |k\rangle \}$

thus $P = \sum_{i=1}^{k} |i\rangle\langle i|$ (It can be shown that $P = P^{\dagger}$)

P is called projector on W. W is thus known as 'vector space' P.

There exists orthogonal complement of P, known as Q

$$Q = I - P$$

it projects onto another vector space $X(F) \subseteq V(F)$ such that

X ~~B~~ $= \{ |k+1\rangle \dots |d\rangle \}$

d) $A: V(F) \to V(F)$ is normal if

$AA^\dagger = A^\dagger A$ $\Rightarrow$ Hermitian operator is normal

<u>Spectral decomposition theorem</u>: An operator is normal operator if and only if it is diagonalizable.

One can prove that Normal matrix is hermitian if eigen values are real. (if A is normal $\to$ it is diagonalisable thus if complex conjugate of T is same $a_i^* = a_i$ thus a_i is real)

$$A = \sum_i a_i |i\rangle\langle i|$$

$$A^\dagger = \sum_i a_i^* |i\rangle\langle i|$$

$$A = A^\dagger \qquad a_i^* = a_i = \text{real}$$

e) Matrix is unitary if

$$U^\dagger U = I$$

They preserve inner products

$$|\alpha\rangle = U|\beta\rangle$$

$$\langle\alpha| = \langle\beta|U^\dagger$$

$$\langle\beta|U^\dagger U|\beta\rangle = \langle\alpha|\alpha\rangle = \langle\beta|\beta\rangle$$

We can find an elegant representation for unitary operator. Say $|v_i\rangle$ are certain orthonormal basis

$$U|v_i\rangle = |\omega_i\rangle$$

thus $\boxed{U = \sum_i |\omega_i\rangle\langle v_i|}$

f) A subclass of Hermitian operators is called positive operators such that (14)

$\langle v | A | v \rangle \geq 0$ (non negative number)

(Any positive operator is automatically a hermitian)

Point 13

Sequence Space: is a broader vector space whose elements are infinite sequence of real or complex numbers.

It can also be supposed (equivalently) to be a functional space whose elements are functions (from natural number to field K)

ℓ^p spaces are the sequence spaces consisting of p-power summable sequences with the p-norm.

$$\sum_n |x_n|^p < \infty$$

$$\|x_n\|_p = \left(\sum_n |x_n|^p\right)^{1/p}$$

ℓ^p is a complete metric space w.r.t the norm. Schwarz inequality and triangl inequalities are followed.

L^p spaces are sequence space consisting of p-power integrable functions (as elements ?)

$$\int_a^b |f(x)|^p dx < \infty$$

L^p $[a, b]$ is defined

For Quantum Mechanics $p = 2$, thus we use L^2 and ℓ^2 spaces consider $L_2(-1,1)$

$$\int_{-1}^{1} dx \, |f(x)|^2 < \infty$$

for any polynomial function (any element in L_2 space)

let $B = \{a_1 x^0, \; a_2 x + b, \; a_3 x^2 + a_4 x + a_5 \cdots\}$ [Basis as monomials]

now lets cook orthonormal basis by using gram-schmidt

$$|e_1\rangle = \frac{a_1 x^0}{\|a_1 x^0\|} \qquad \|a_1 x^0\|^2 = \int_{-1}^{1} a_1^2 \, dx = a_1^2 \, 2$$

$$|e_1\rangle = \frac{1}{\sqrt{2}}$$

$$|e_2\rangle = \frac{a_2 x + b - \left[\int_{-1}^{1} \frac{1}{\sqrt{2}} (a_2 x + b) \, dx\right] \frac{1}{\sqrt{2}}}{\| \cdot \|}$$

$$= \frac{a_2 x}{\|a_2 x\|} = \sqrt{\frac{3}{2}} \, x$$

now consider $L_2(-\infty, \infty)$

normalisation factor

there $$\int_{-\infty}^{\infty} d\mu(x) \, Q_n^*(x) \, Q_m(x) = A_n \delta_{nm}$$

measure that ensures the finiteness

$$d\mu(x) = e^{-x^2} dx \qquad \{\text{Hermite polynomials}\}$$

for $L_2(0, \infty)$

$$d\mu(x) = e^{-x} dx \qquad \{\text{Laguerre polynomials}\}$$

consider the fourier transforms

$$f(x+2L) = f(x)$$

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=0}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where $\cos\left(\frac{n\pi x}{L}\right)$ and $\sin\left(\frac{n\pi x}{L}\right)$ form orthogonal basis vectors

also this equation can be converted to

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\left(\frac{in\pi x}{L}\right)}$$

C_n is the scalar $\in \mathbb{C}$ and $e^{\frac{in\pi x}{L}}$ are basis vectors

$$C_n = \langle e_n | \psi \rangle$$

thus $C_n = \int_0^{2L} e^{-\frac{in\pi x}{L}} f(x)\, dx$

for non periodic $f(x)$ $(-\infty, \infty)$

$$\boxed{f(x) = \int_{-\infty}^{\infty} \tilde{f}(k)\, e^{ikx}\, dk}$$

Let there be L_2 space $_{[a,b]}$ with $f_n(x)$ $[n = 1, 2 \cdots]$ (countable functions)

Then orthonormality $\Rightarrow \int_a^b f_n^*(x) f_m(x)\, dx = \delta_{nm}$ $\left[\langle e_n | e_m \rangle = \delta_{nm}\right]$

completeness $\Rightarrow \sum_{n=1} f_n(x) f_n^*(x') = \delta(x-x')$ $\left[\sum_n |e_n\rangle\langle e_n| = I\right]$

<u>justification of completeness</u>

we have to find an operator which when operated on the function produces the function itself. now $\delta(x-x')$ is used $x, x' \in (a,b)$

$$\int_a^b \delta(x-x')\, g(x')\, dx' = g(x)$$

from the definition we can see that

$$\sum_n f_n(x) \int_a^b f_n^*(x') \, g(x') \, dx' = \sum_n f_n(x) \, C_n = g(x)$$

thus

$$\int_a^b \sum_n f_n(x) \, f_n^*(x') \, g(x') \, dx' = \int_a^b \delta(x-x') \, g(x') \, dx'$$

$$\boxed{\delta(x-x') = \sum_n f_n(x) \, f_n^*(x')}$$

The operations of operator is defined by

$$L \, Q(x) = \int_a^b K(x,x') \, Q(x') \, dx'$$

$K(x,x')$ is kernel

Define Adjoint of operator as

$$(\psi, LQ) = (L^{adj} \psi, Q)$$

$$\int_a^b \psi^*(x) \, LQ(x) \, dx = \int_a^b \psi^*(x) \int_a^b K(x,x') \, Q(x') \, dx' \, dx$$

$$= \int_a^b Q(x') \int_a^b \psi^*(x) \, K(x,x') \, dx \, dx'$$

$$\text{RHS} = \int_a^b \left(L^{adj} \psi(x)\right)^* Q(x) \, dx = \int_a^b Q(x) \left[\int_a^b F(x,x') \, \psi(x') \, dx'\right]^* dx$$

$$= \int_a^b Q(x) \left[\int_a^b F^*(x,x') \, \psi^*(x') \, dx'\right] dx$$

$$F^*(x,x') = K(x',x)$$

$$L^{adj} = \int_a^b dx' \, K^*(x',x)$$